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Representation in L_p by Series of Translates and Dilates of One Function

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We study minimal conditions under which the function system of dyadic translates and dilates of one fixed function $\varphi(t)$ with support in $[0, 1]$ forms a representation system in $L_p(0, 1)$, i.e., that any function $f(t) \in L_p(0, 1)$ can be represented by at least one L_p-convergent series with respect to this system. Generalizations to the situation of a multiresolution analysis on $\mathbf{R}^n$ are also discussed. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let be given a function $\varphi(t)$ with support in $[0, 1]$, and consider the system

$$\varphi_{k,i}(t) = \varphi(2^k t - i), \qquad i = 0, ..., 2^k - 1; \quad k = 0, 1, 2, ...$$

We are going to study minimal conditions under which this system (or a subsystem of it) is a representation system in $L_p(0, 1)$ for some $0 < p < \infty$,

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i.e., whether for any $f \in L_p(0, 1)$ there exists at least one L_p-convergent series representation:

$$f(t) = \sum_{k=0}^{\infty} \sum_{i=0}^{2^k - 1} a_{k,i} \varphi_{k,i}(t).$$

The notion of representation systems, which generalizes the notion of a basis, was introduced by A. A. Talaljan [T1] but arised already in connection with the classical investigations by D. E. Menshov [Me] on the representation of arbitrary measurable functions by trigonometric series. There is a number of results, both on representation systems in spaces without bases (such as L_p, $0<p<1$), and on cases where some classical system does not form a basis in a particular space. For instance, A. A. Talaljan [T1, T2] showed that any complete orthonormal system in $L_2(0, 1)$ forms a representation system in $L_p(0, 1)$, $0<p<1$. Moreover, this property remains true even if a finite number of functions are deleted from the orthonormal system. As a consequence, if we take

$$\varphi(t) = \begin{cases} -1, & t \in (\frac{1}{2}, 1] \\ 1, & t \in (0, \frac{1}{2}] \end{cases},$$

then $\{\varphi_{k,i}\}$ is a representation system in $L_p(0, 1)$, $0<p<1$ (to this end, consider the Haar system and delete the first (constant) function). Clearly, for $p \geq 1$ this is not true, a constant $\neq 0$ can not be represented. Another result we want to mention is as follows (P. L. Uljanov [U]): The system $\{\varphi_{k,i}\}$ with the generating function

$$\varphi(t) = \begin{cases} 1-2t, & \frac{1}{2} < t \leq 1 \\ 2t, & 0 \leq t \leq \frac{1}{2} \end{cases}$$

(which is actually the classical Faber-Schauder-System with the first two functions deleted) forms a representation system in $L_p(0, 1)$, $0<p<\infty$. There are investigations on subsystems of representation systems [I, F1, F2], on representation systems in $\phi(L)$ [U, I, O1, O2, F1, F2], on the representation of complex functions by series of exponentials [K] etc.

In Section 2 we prove the following result which generalizes the above examples in a rigorous way.

THEOREM 1. (a) *Let $\varphi \in L_q(0, 1)$ for some $1 \leq q < \infty$. If*

$$\int_0^1 \varphi(t)\, dt \neq 0, \qquad (*)$$

then $\{\varphi_{k,i}\}$ is a representation system in $L_p(0, 1)$ for any $0<p \leq q$.

(b) *Let* $0 \neq \varphi \in L_2(0,1)$. *Then* $\{\varphi_{k,i}\}$ *is a representation system in* $L_p(0,1)$, $0<p<1$.

Obviously, this result completely solves the question of $\{\varphi_{k,i}\}$ being a representation system in $L_p(0,1)$ for $1 \leqslant p < \infty$: $\varphi(t) \in L_p(0,1)$ and $(*)$ are necessary and sufficient conditions in this case. For $p<1$, a final answer is still missing.

The method we use is elementary. The crucial Lemma 1 of Section 2 shows the existence of a constant $\lambda_0 \neq 0$ such that

$$\int_0^1 |1-\lambda_0 \varphi(t)|^p \, dt < 1. \tag{$**$}$$

From this simple fact, we can construct L_p-convergent series with respect to $\{\varphi_{k,i}\}$ for any $f \in L_p(0,1)$. The construction shows that the systems under consideration never form bases: one can find many representations for any given function as well as delete functions from $\{\varphi_{k,i}\}$ without destroying the representation property. We also give a necessary and sufficient condition on a subsystem of $\{\varphi_{k,i}\}$ to remain still a representation system in L_p.

The interest in systems of the above type which are generated by translation and dilation from one function $\varphi(t)$ stems also from the recent research activities on multiresolution analysis and wavelets where questions of approximation and representation by analogous systems on $\mathbf{R}^n$ have been studied to a certain generality, cf. [D, BDR, JM]. We address this case of representation systems $\{\varphi_{k,i}\}$ in $L_p(\mathbf{R}^n)$ in Section 3, allowing also some generating functions φ with noncompact support.

2. Representation Systems in $L_p[0,1]$

Let $\varphi(t)\colon [0,1] \to \mathbf{R}$ be an arbitrary measurable function which is extended outside $[0,1]$ by zero. We define the system $\{\varphi_{k,i}\}$ of dyadic translates and dilates of φ on $[0,1]$ by

$$\varphi_{k,i}(t) = \varphi(2^k t - i), \qquad t \in [0,1]; \quad k=0,1,...; \quad i=0,...,2^k-1.$$

Denote (as in the classical case of the Haar system)

$$\varphi_n(t) \equiv \varphi_{k,i}(t), \qquad n = 2^k + i, \quad k=0,1,..., \quad i=0,...,2^k-1.$$

Let $I_n \equiv I_{k,i} = (i/2^k, i+1/2^k)$ stand for the dyadic interval related to φ_n.

Concerning the L_p-spaces $(0<p<\infty)$ we introduce the following notation:

$$\|f\|_{L_p} = \left(\int_0^1 |f(t)|^p \, dt \right)^{1/\tilde{p}}, \qquad \tilde{p} = \max(1,p),$$

denotes the usual norm of a function $f \in L_p \equiv L_p(0, 1)$ if $1 \leqslant p < \infty$, and generates the L_p-metric if $0 < p < 1$. Obviously, accepting this we can use the triangle inequality

$$\|f+g\|_{L_p} \leqslant \|f\|_{L_p} + \|g\|_{L_p}, \qquad f, g \in L_p,$$

for all $0 < p < \infty$. The same notation carries over to L_p-spaces on general domains in $\mathbf{R}^n$.

DEFINITION [T1, T2]. A system of $\{f_n\}_{n=1}^{\infty} \subset L_p$, $0 < p < \infty$ is called a representation system in the space L_p if for any $f \in L_p$ there exists a series $\sum_{k=1}^{\infty} c_k f_k$ such that

$$\lim_{n \to \infty} \left\| f - \sum_{k=1}^{n} c_k f_k \right\|_{L_p} = 0.$$

This definition generalizes to F-spaces.

THEOREM 2. (a) *Let φ satisfy the assumptions of Theorem* 1(a). *Then a subsystem $\{\varphi_{n_l}\}$ of the system $\{\varphi_n\}$ is a representation system in L_p, $0 < p \leqslant q$, if and only if*

$$\forall N \in \mathbf{N} \qquad \operatorname{mes}\left\{ \bigcup_{l=N}^{\infty} I_{n_l} \right\} = 1. \tag{1}$$

(b) *If φ satisfies the assumptions of Theorem* 1(b) *then a subsystem $\{\varphi_{n_l}(t)\}$ is a representation system in L_p, $0 < p < 1$, if and only if* (1) *is fulfilled.*

We first prove the following lemma.

LEMMA 1. *Under the assumptions of Theorem* 1(a) *resp.* (b) *there exists a constant $\lambda_0 \neq 0$ such that*

$$\sigma_0 \equiv \|1 - \lambda_0 \varphi\|_{L_p} < 1 \tag{2}$$

Proof. We start with case (a) of Theorem 1. Let $\varphi \in L_q$, $q \geqslant 1$, satisfy $\int_0^1 \varphi(t)\, dt = \delta > 0$ (if $\delta < 0$ then consider $-\varphi(t)$).

Obviously, it is sufficient to prove the estimate for $p = q$ since by the Hölder inequality $\|g\|_{L_p} \leqslant \|g\|_{L_q}^{\min(p,1)}$ for all $g \in L_q$ and $p < q$. For $p \geqslant 1$ we have the inequalities

$$|1 - x|^p \leqslant 1 - px + c_0 x^2, \qquad |x| \leqslant \tfrac{1}{2},$$

$$|1 - x|^p \leqslant 1 + c_1 |x| + c_2 |x|^p, \qquad x \in \mathbf{R}.$$

which hold with some positive constants c_0, c_1, c_2. Let

$$E_\alpha = \left\{t \colon |\varphi(t)| \leqslant \frac{1}{2\alpha}\right\}, \qquad E_\alpha^* = \left\{t \colon |\varphi(t)| > \frac{1}{2\alpha}\right\}. \quad \alpha > 0.$$

We use the first inequality on E_α, the second one on E_α^*. For $0 < \lambda \leqslant \alpha$ we obtain

$$\begin{aligned}
\|1 - \lambda\varphi\|_{L_p}^p &= \int_{E_\alpha} |1 - \lambda\varphi(t)|^p \, dt + \int_{E_\alpha^*} |1 - \lambda\varphi(t)|^p \, dt \\
&\leqslant \operatorname{mes} E_\alpha - p \int_{E_\alpha} \lambda\varphi(t) \, dt + c_0\lambda^2 \int_{E_\alpha} \varphi^2(t) \, dt \\
&\quad + \operatorname{mes} E_\alpha^* + c_1\lambda \int_{E_\alpha^*} |\varphi(t)| \, dt + c_2\lambda^p \int_{E_\alpha^*} |\varphi(t)|^p \, dt \\
&\leqslant 1 - p\lambda\delta + c_3\lambda \int_{E_\alpha^*} |\varphi(t)| \, dt + c_2\lambda^p \int_{E_\alpha^*} |\varphi(t)|^p \, dt + \frac{c_0\lambda^2}{4\alpha^2} \equiv T(\lambda).
\end{aligned}$$

Since $\operatorname{mes} E_\alpha^* \to 0$ for $\alpha \to 0_+$ and $\varphi \in L_p$, $p = q \geqslant 1$, one can now fix $\alpha_0 > 0$ such that $T'(0_+) < 0$ which together with $T(0) = 1$ implies the existence of $0 < \lambda_0 < 1/2\alpha_0$ with the desired properties. This proves the assertion in the case (a) of Theorem 1.

We come now to the assumptions (b) of Theorem 1. Let $0 < p < 1$. By the Taylor formula there exists a constant $c > 0$ such that

$$|1 - x|^p \leqslant 1 - px - cx^2, \qquad |x| \leqslant \tfrac{1}{2}.$$

Let

$$G(\lambda) = \int_0^1 |1 - \lambda\varphi(t)|^p \, dt, \qquad \lambda \in \mathbf{R},$$

and

$$E_\lambda = \{t \colon |\lambda\varphi(t)| \leqslant \tfrac{1}{2}\} = E_{-\lambda}, \qquad E_\lambda^* = \{t \colon |\lambda\varphi(t)| > \tfrac{1}{2}\} = E_{-\lambda}^*.$$

Then, using the above inequality on E_λ and the triangle inequality on E_λ^*, we obtain

$$G(\lambda) = \int_{E_\lambda} + \int_{E_\lambda^*} \leqslant 1 - \lambda p \int_{E_\lambda} \varphi(t) \, dt - c \int_{E_\lambda} |\lambda\varphi(t)|^2 \, dt + \int_{E_\lambda^*} |\lambda\varphi(t)|^p \, dt.$$

For $t \in E_\lambda^*$ we have $|\lambda\varphi(t)|^p \leqslant 2^{2-p} \, |\lambda\varphi(t)|^2$. Thus, we get

$$\frac{G(\lambda) + G(-\lambda)}{2} \leqslant 1 - \lambda^2 \left[c \int_{E_\lambda} |\varphi(t)|^2 \, dt - 2^{2-p} \int_{E_\lambda^*} |\varphi(t)|^2 \, dt \right].$$

It can easily be seen that for $\lambda \to 0$

$$\int_{E_\lambda} |\varphi(t)|^2\, dt \to \|\varphi\|_{L_2}^2 > 0,$$

and that

$$\int_{E_\lambda^*} |\varphi(t)|^2\, dt \to 0.$$

Hence there exists a $\lambda_0 > 0$ such that $(G(\lambda_0) + G(-\lambda_0))/2 < 1$ which gives the result for case (b).

The following discussion shows that the inequality (2) is all what we need to prove the assertions of Theorems 1 and 2. For brevity, denote $g_l(x) = \lambda_0 \varphi_{n_l}(x)$ where λ_0 is taken from Lemma 1, and $\{\varphi_{n_l}\}_{l=1}^{\infty}$ is any subsystem of the system $\{\varphi_n\}$. A function S will be called dyadic step function if, for some $k \in \mathbf{N}$, S is constant on all intervals $I_{k,i}$, i.e., if

$$S(t) = \sum_{i=0}^{2^k - 1} \lambda_{k,i} \chi_{I_{k,i}}(t)$$

where $\chi_I(t)$ denotes the characteristic function of an interval I, and the $\lambda_{k,i}$ are any real numbers.

LEMMA 2. *Assume that $\varphi \in L_p(0, 1)$ satisfies* (2), *and that the subsystem $\{\varphi_{n_l}\}$ satisfies* (1). *Fix some $\sigma \in (\sigma_0, 1)$. Then for any step function S, and arbitrary $N \in \mathbf{N}$ there exists a finite sum $h \equiv \sum_{l=N}^{M} c_l g_l$, $M > N$, such that*

$$\|S - h\|_{L_p} \leqslant \sigma \|S\|_{L_p} \tag{3}$$

$$\left\| \sum_{l=N}^{n} c_l g_l \right\|_{L_p} < (1 + \sigma) \|S\|_{L_p}, \qquad N \leqslant n \leqslant M. \tag{4}$$

Proof. Consider a dyadic step function $S \not\equiv 0$ as given above (with a integer k fixed). According to (1) and the obvious properties of dyadic intervals, we can find a subsequence of indices $\max(N, 2^k) \leqslant l_1 < l_2 < \cdots < l_j < \cdots$ such that the intervals $E_j \equiv I_{n_{l_j}}$ are pairwise disjoint, and that still mes $\bigcup_j E_j = 1$. By construction, each E_j belongs to exactly one $I_{k,i}$, and we set $\lambda_j = \lambda_{k,i}$.

We can now check that

$$h = \sum_{j=1}^{m} \lambda_j g_{l_j}$$

has the desired properties for sufficiently large m (to fit the notation used in the above formulation of Lemma 2, set $c_l=\lambda_j$ if $l=l_j$, and $c_l=0$ otherwise). Let $\tilde{E}_m=[0,1]\setminus\bigcup_{j=1}^m E_j$. Obviously, by this construction and by (2), we get

$$\begin{aligned}\|S-h\|_{L_p}^p &= \int_{\tilde{E}_m} |S(t)|^p\,dt + \int_0^1 |1-\lambda_0\varphi(t)|^p\,dt \sum_{j=1}^m |\lambda_j|^p \operatorname{mes} E_j \\ &\leqslant \int_{\tilde{E}_m} |S(t)|^p\,dt + \sigma_0^p\,\|S\|_{L_p}^p,\end{aligned}$$

where λ_0 and σ_0 are given in Lemma 1. Since mes $\tilde{E}_m \to 0$ for $m\to\infty$, the remaining integral over $\tilde{E}_m$ will be arbitrarily small. This establishes (3) if we fix some sufficiently large m $(=M)$. Since the intervals E_j are disjoint and supp $g_{l_j}\subset E_j$, we have

$$\left\|\sum_{j=1}^n \lambda_j g_{l_j}\right\|_{L_p} \leqslant \|h\|_{L_p} \leqslant \|h-S\|_{L_p} + \|S\|_{L_p} \leqslant (1+\sigma)\,\|S\|_{L_p}$$

for all $n\leqslant m$ which finishes the proof of Lemma 2.

Proof of Theorem 2. We use an induction argument. Let $f_0=f$, $N_0=M_0=0$. In the induction step, for given f_{r-1} and M_{r-1}, we first define some dyadic step function S_r such that

$$\|f_{r-1}-S_r\|_{L_p}\leqslant 2^{-r-1}.$$

After this, by Lemma 2 applied to this S_r and some $N_r>M_{r-1}$ we find a linear combination

$$h_r=\sum_{l=N_r}^{M_r} c_l g_l$$

such that

$$\begin{aligned}&\|S_r-h_r\|_{L_p}\leqslant\sigma\,\|S_r\|_{L_p},\\ &\left\|\sum_{l=N_r}^n c_l g_l\right\|_{L_p}\leqslant(1+\sigma)\,\|S_r\|_{L_p}, \qquad n=N_r,\ldots,M_r.\end{aligned}$$

for some fixed $\sigma_0<\sigma<1$. Finally, to finish the induction step, we set $f_r=f_{r-1}-h_r$.

To prove the theorem, we will check that the series

$$\sum_{r=1}^\infty h_r\equiv\sum_{l=1}^\infty c_l g_l\equiv\sum_{l=1}^\infty \lambda_0 c_l\varphi_{n_l}$$

represents f in L_p (we put $c_l=0$ for the remaining indices l). To this end, for arbitrarily given $n>0$, define the index $r\geqslant 1$ such that $M_{r-1}\leqslant n<M_r$. Then, by the above construction,

$$\begin{aligned}\left\|f-\sum_{l=1}^{n} c_l g_l\right\|_{L_p} &\leqslant \|f_{r-1}\|_{L_p}+\left\|\sum_{l=N_r}^{n} c_l g_l\right\|_{L_p}\\ &\leqslant \|f_{r-1}\|_{L_p}+(1+\sigma)\,\|S_r\|_{L_p}\\ &\leqslant (2+\sigma)\,\|f_{r-1}\|_{L_p}+(1+\sigma)\,\|f_{r-1}-S_r\|_{L_p}\\ &\leqslant 2^{-r}+3\,\|f_{r-1}\|_{L_p}.\end{aligned}$$

Note that for $n<N_r$ the second term may be neglected. Since

$$\begin{aligned}\|f_r\|_{L_p} &\leqslant \|f_{r-1}-S_r\|_{L_p}+\|S_r-h_r\|_{L_p}\\ &\leqslant 2^{-r-1}+\sigma\,\|S_r\|_{L_p}\leqslant 2^{-r}+\sigma\,\|f_{r-1}\|_{L_p},\end{aligned}$$

we get recursively

$$\begin{aligned}\|f_r\|_{L_p} &\leqslant 2^{-r}+2^{-r+1}\sigma+\cdots+2^{-1}\sigma^{r-1}+\sigma^r\,\|f\|_{L_p}\\ &\leqslant r(\max(2^{-1},\sigma)^r+\sigma^r\,\|f\|_{L_p},\end{aligned}$$

which finally shows the convergence of the series to f. The proof of the sufficiency of (1) for the assertion of Theorem 2 is now complete.

The necessity is obvious: if (1) is violated then there exists a set $E\subset[0,1]$ of positive measure such that all φ_{n_l} but a finite number vanish on E. Therefore, it is easy to construct a function $f\in L_p[0,1]$ with support in E which is not in the L_p closure of the given subsystem.

Remark 1. Since the whole system $\{\varphi_n\}$ obviously satisfies (1), Theorem 1 is a consequence of Theorem 2.

Remark 2. It can be shown that condition (1) of Theorem 2 can be replaced by a condition formulated directly in terms of the functions φ_{n_l}:

$$\forall\varepsilon>0\quad \forall N\in\mathbf{N}\quad \exists m>N\colon \operatorname{mes}\left\{t\colon \sum_{l=N}^{m}|\varphi_{n_l}(t)|\neq 0\right\}>1-\varepsilon.$$

Remark 3. One easily observes from the proofs that Theorem 1 carries over to the spaces $L_p([0,1]^n)$, $p>0$, or even to L_p spaces on arbitrary measurable sets $\Omega\subset\mathbf{R}^n$, $n\geqslant 1$. The underlying construction then starts with a L_p function $\varphi\neq 0$ with compact support, and the system is defined by all those

$$\varphi_{k,\mathbf{i}}(\mathbf{x})=\varphi(2^k\mathbf{x}-\mathbf{i}),\qquad \mathbf{x}\in\mathbf{R}^n,\quad \mathbf{i}\in\mathbf{Z}^n,\quad k\in\mathbf{Z}.$$

that do not vanish on a set of positive measure in Ω. Along the same lines, Theorem 2 may be generalized.

Remark 4. As was mentioned above, if we are restricted to the classical situation $1 \leqslant p < \infty$, the conditions $\varphi \in L_p$ and $(*)$ are necessary and sufficient for $\{\varphi_n\}$ to form a representation system in L_p. However, in the first example given in the Introduction where $(*)$ is violated it suffices to add a single constant function to the system and one arrives at a representation system (in this case, we have the Haar system which is even a Schauder basis in L_p). One might ask whether there is a general possibility to repair the systems where $(*)$ does not hold by adding a finite number of auxiliary functions. A simple example shows that this is not the case: If φ has mean value zero on each dyadic interval of the form $[2^{-k-1}, 2^{-k}]$, $k=0, 1, ...$, then any function from the corresponding system $\{\varphi_n\}$ is L_2 orthogonal to the subsystem of all Haar functions with index $n = 2^k$, $k = 0, 1, ...$, which span an infinite-dimensional subspace in L_p $(1 \leqslant p < \infty)$.

For $p < 1$, the condition $(*)$ seems to be no more important (compare the result of Theorem 1(b)). However, growth conditions may come in. As we learned from G. Tachev, the crucial property $(**)$ is not satisfied for the functions $\varphi(t) = t^{-\beta}$ if $2/(p+1) \leqslant \beta < 1/p$, $0 < p < 1$.

3. Representation Systems in $L_p(R^n)$

The present section is motivated by the recent investigations on multiresolution analysis, shift-invariant subspaces, and wavelet constructions on $\mathbf{R}^n$. Throughout this section, let $\varphi(\mathbf{t}) \in L_p \equiv L_p(\mathbf{R}^n)$, with $n \geqslant 1$ and $1 \leqslant p < \infty$ be given, and define (as in Remark 3)

$$\varphi_{k,\mathbf{i}}(\mathbf{t}) = \varphi(2^k \mathbf{t} - \mathbf{i}), \qquad t \in \mathbf{R}^n, \quad \mathbf{i} \in \mathbf{Z}^n, \quad k \in \mathbf{Z}.$$

Denote by

$$V_k(\varphi) = \overline{\operatorname{span}\{\varphi_{k,\mathbf{i}} : \mathbf{i} \in \mathbf{Z}^n\}}\,|_{L_p}, \qquad k \in \mathbf{Z}$$

the sequence of dyadic (with respect to $h = 2^{-k}$) principal shift-invariant subspaces corresponding to φ (see [BDR] for some generalities and history). Formally, $\{V_k(\varphi)\}$ looks like a multiresolution analysis ([Ml]; [D], Chapter 5; [JM]) but we will not assume that this sequence of closed subspaces of L_p is increasing which is a basic assumption in much of the wavelet literature.

The question we will discuss here is whether $\{\varphi_{k,\mathbf{i}}\}$ forms a representation system in L_p. If the answer is yes, as a by-product we get

$$\left.\overline{\sum_k V_k(\varphi)}\right|_{L_p} = L_p.$$

If $V_k(\varphi) \subset V_{k+1}(\varphi)$ the sum may be replaced by the union of the subspaces. The density in L_p of the latter set which is one of the basic assumptions of a multiresolution analysis ($p=2$) has been studied to a certain generality in [D] (see Proposition 5.3.2 and the remarks on pp. 143–145), [Md], [JM] (Theorem 2.5), [JL] under various assumptions on φ (as a rule, these papers require $V_k(\varphi) \subset V_{k+1}(\varphi)$ but see [BDR] (Theorem 1.7)).

In Section 2, Remark 3, we have already stated that in the case of a compactly supported generating function φ the result of Theorem 1 can be carried over to the present situation. In addition, in this case the summation order of the constructed series representation does not matter. In the following, we will call a series with respect to $\{\varphi_{k,\mathbf{i}}\}$ unconditionally L_p-convergent if any (linear) ordering of the index set $\{(k, \mathbf{i})\}$ leads to an L_p-convergent series, with the same limit $f \in L_p$.

We will now state a sufficient condition for $\{\varphi_{k,\mathbf{i}}\}$ to form an unconditional representation system in L_p (i.e., the representation we can find for any $f \in L_p$ will be unconditionally L_p-convergent to f) which also covers some φ with noncompact support but still requires certain additional decay properties for $p > 1$.

THEOREM 3. *Let $\varphi \in L_p$ for some $1 \leq p < \infty$, for $1 < p < \infty$ we additionally require*

$$|\varphi(\mathbf{t})| \leq C \cdot |\mathbf{t}|^{-n-\gamma}, \qquad |\mathbf{t}| \to \infty$$

with some $\gamma > 0$. Suppose

$$\int_{\mathbf{R}^n} \varphi(\mathbf{t})\, d\mathbf{t} = \hat{\varphi}(0) \neq 0.$$

Then $\{\varphi_{k,\mathbf{i}}\}$ is an unconditional representation system in L_p.

Proof. The main idea is first to prove an analog of Lemma 2. Without loss of generality, let $\hat{\varphi}(0) = 1$. Thus, for any sufficiently large cube $W = (-2^r, 2^r)^n$ defined by a natural number r

$$\tfrac{1}{2} < \delta \equiv \int_W \varphi(\mathbf{t})\, d\mathbf{t} < \tfrac{3}{2}.$$

The value of r will be fixed below.

From now on we consider only the subsystem

$$\varphi_{k,\mathbf{i}}(\mathbf{t}) \equiv \varphi_{k,2^{r+1}\mathbf{i}}(\mathbf{t})$$

which depends on the choice of W and, thus, on r. To each $\psi_{k,\mathbf{i}}$ there corresponds its cube $W_{k,\mathbf{i}}$ of sidelength 2^{r+1-k} (the shifted and dilated $W = W_{0,\mathbf{0}}$), and the collection

$$\mathscr{R}_k \equiv \{W_{k,\mathbf{i}} : \mathbf{i} \in \mathbf{Z}^n\}$$

forms a partition of $\mathbf{R}^n$ into non-intersecting (open) cubes for arbitrary $k \in \mathbf{Z}$.

Let S_k denote any step function with respect to $\mathscr{R}_k$, i.e.

$$S_k(\mathbf{t}) = \lambda_{k,\mathbf{i}}, \qquad \mathbf{t} \in W_{k,\mathbf{i}}, \quad \mathbf{i} \in \mathbf{Z}^n.$$

Obviously, $S_k \in L_p$ iff

$$\|S_k\|_{L_p}^p = 2^{(r+1-k)n} \sum_{\mathbf{i} \in \mathbf{Z}^n} |\lambda_{k,\mathbf{i}}|^p < \infty.$$

The above mentioned analog of Lemma 2 we are going to prove reads as follows:

LEMMA 3. *In the above construction, one can fix r and find some reals $\lambda_0 \neq 0$ and $\sigma \in (0, 1)$ such that (independently of S_k and k)*

$$\left\| S_k - \lambda_0 \sum_{\mathbf{i} \in \mathbf{Z}^n} \lambda_{k,\mathbf{i}} \psi_{k,\mathbf{i}} \right\|_{L_p} \leqslant \sigma \|S_k\|_{L_p}. \tag{5}$$

Proof. It suffices to consider $k = 0$, we therefore omit the index k for brevity. Then

$$\left\| S - \lambda \sum_{\mathbf{i} \in \mathbf{Z}^n} \lambda_{\mathbf{i}} \psi_{\mathbf{i}} \right\|_{L_p} \leqslant \overbrace{\left(\sum_{\mathbf{i} \in \mathbf{Z}^n} \int_{W_{\mathbf{i}}} |\lambda_{\mathbf{i}}(1 - \lambda\psi_{\mathbf{i}}(\mathbf{t}))|^p \, d\mathbf{t} \right)^{1/p}}^{\equiv A}$$

$$+ \lambda \overbrace{\left(\sum_{\mathbf{i} \in \mathbf{Z}^n} \int_{W_{\mathbf{i}}} \left| \sum_{\mathbf{j} \neq \mathbf{i}} \lambda_{\mathbf{j}} \psi_{\mathbf{j}}(\mathbf{t}) \right|^p d\mathbf{t} \right)^{1/p}}^{\equiv B}$$

To the second term we apply the technique behind Theorem 2.1 from [JM]:

$$B=\int_W \sum_{\mathbf{m}\in \mathbf{Z}^n} \left| \sum_{\mathbf{j}\neq \mathbf{0}} \lambda_{\mathbf{j}+\mathbf{m}} \psi_{\mathbf{j}}(\mathbf{t}) \right|^p dt$$

$$\leqslant \int_W \sum_{\mathbf{m}\in \mathbf{Z}^n} \left\{ \underbrace{\left(\sum_{\mathbf{j}\neq \mathbf{0}} |\psi_{\mathbf{j}}(\mathbf{t})| \right)}_{\equiv \Phi(\mathbf{t})}^{p-1} \sum_{\mathbf{j}\neq \mathbf{0}} |\lambda_{\mathbf{j}+\mathbf{m}}|^p \, |\psi_{\mathbf{j}}(\mathbf{t})| \right\} dt$$

$$= \int_W \Phi(\mathbf{t})^{p-1} \sum_{\mathbf{j}\neq \mathbf{0}} \sum_{\mathbf{m}\in \mathbf{Z}^n} |\lambda_{\mathbf{j}+\mathbf{m}}|^p \, |\psi_{\mathbf{j}}(\mathbf{t})| \, dt$$

$$= \left(\sum_{\mathbf{i}\in \mathbf{Z}^n} |\lambda_{\mathbf{i}}|^p \right) \int_W \Phi(\mathbf{t})^p \, dt = 2^{-(r+1)n} \, \|S\|_{L_p}^p \int_W \Phi(\mathbf{t})^p \, dt.$$

The first term of the above expression can be transformed into

$$A = \left(\sum_{\mathbf{i}\in Z^n} |\lambda_{\mathbf{i}}|^p \right) \underbrace{\int_W |1-\lambda\varphi(\mathbf{t})|^p \, dt}_{\equiv \beta_p(\lambda)} = 2^{-(r+1)n} \, \|S\|_{L_p}^p \, \beta_p(\lambda).$$

$\beta_p(\lambda)$ can be estimated along the lines of Lemma 1. Consider first the simplest case $p=1$. Here,

$$\beta_1(\lambda) = \int_W (1-\lambda\varphi(\mathbf{t}) + 2\max\{0, \lambda\varphi(\mathbf{t})-1\}) \, dt$$

$$\leqslant 2^{(r+1)n} - \lambda\delta + 2\lambda \int_{\mathbf{t}:\, \varphi(\mathbf{t}) \geqslant 1/\lambda} \varphi(\mathbf{t}) \, dt$$

$$= 2^{(r+1)n} \left(1 - \lambda 2^{-(r+1)n} \left(\delta - 2 \int_{\mathbf{t}:\, \varphi(\mathbf{t}) \geqslant 1/\lambda} \varphi(\mathbf{t}) \, dt \right) \right).$$

On the other hand, for $\varphi \in L_1$ we have

$$\int_W \Phi(\mathbf{t}) \, dt = \int_{\mathbf{R}^n \setminus W} |\varphi(\mathbf{t})| \, dt \to 0, \qquad r \to \infty.$$

Fix a sufficiently large r such that this integral is less than 1/8. Then the substitution into the above inequalities yields

$$\left\| S - \lambda \sum_{\mathbf{i}\in \mathbf{Z}^n} \lambda_{\mathbf{i}} \psi_{\mathbf{i}} \right\|_{L_1}$$

$$\leqslant \left(1 - \lambda 2^{-(r+1)n} \cdot \left(\delta - 2 \int_{\mathbf{t}:\, \varphi(\mathbf{t}) \geqslant 1/\lambda} \varphi(\mathbf{t}) \, dt - 1/8 \right) \right) \|S\|_{L_1}.$$

Now we can finish the argument. Choose $\lambda = \lambda_0$ such that the integral in the above formula is also bounded by 1/8 (it tends to zero if $\lambda \to 0_+$). Since $\delta > 1/2$ we get the desired result with $\sigma = 1 - \lambda_0 2^{-(r+1)n-3}$.

For $p > 1$ one may use the inequalities

$$|1-x|^p \leqslant 1 - px + c\,|x|^p, \qquad x \in \mathbf{R}, \quad 1 < p < 2,$$

resp.

$$|1-x|^p \leqslant 1 - px + c(x^2 + |x|^p), \qquad x \in \mathbf{R}, \quad 2 \leqslant p < \infty,$$

which can be checked by simple calculus. This gives in the same way

$$2^{-(r+1)n} \beta_p(\lambda) \leqslant 1 - 2^{-(r+1)n}(p\lambda\delta - c\lambda^p \, \|\varphi\|_{L_p})$$

for $1 < p < 2$, for $p \geqslant 2$ a further term $c\lambda^2 \, \|\varphi\|_{L_2}$ has to be added correspondingly.

To estimate the integral involving $\mathbf{\Phi}$ we make use of the decay property. For sufficiently large r we obtain

$$\int_W \mathbf{\Phi}(\mathbf{t})^p \, d\mathbf{t} \leqslant C 2^{(r+1)n} \cdot \left(\sum_{\mathbf{i} \neq \mathbf{0}} (2^r \, |\mathbf{i}|)^{-(n+\gamma)} \right)^p \leqslant C 2^{-r((n+\gamma)p-n)}.$$

Putting things together, we arrive at

$$\left\| S - \lambda \sum_{\mathbf{i} \in \mathbf{Z}^n} \lambda_{\mathbf{i}} \psi_{\mathbf{i}} \right\|_{L_p}$$

$$\leqslant \|S\|_{L_p} \left((1 - 2^{-(r+1)n}(p\lambda\delta - c\lambda^p \, \|\varphi\|_{L_p}))^{1/p} + C 2^{-r(n+\gamma)} \lambda \right)$$

for $\lambda \geqslant 0$ and $1 < p < 2$ (the case $p \geqslant 2$ is completely analogous). Since $\gamma > 0$ we can take r sufficiently large such that the first derivative of this upper bound at $\lambda = 0_+$ is negative. With this r fixed, we can now find the desired λ_0. This proves (5).

With this substitute for Lemma 2 at hand, we can finish the proof of Theorem 3 along the lines of the recursive construction used for Theorem 2. Note that the analog of (4) trivially follows from (5):

$$\left\| \lambda_0 \sum_{\mathbf{i} \in K} \lambda_{k,\mathbf{i}} \psi_{k,\mathbf{i}} \right\|_{L_p} \leqslant (1+\sigma) \, \|S_k\|_{L_p}, \qquad \forall k \subset \mathbf{Z}^n. \tag{6}$$

The dyadic step functions are chosen such that they fit the assumptions of Lemma 3, the functions h_r are now explicitly given by the expression in (5). The unconditional convergence of the whole series easily follows from (6) and the geometric decay of $\|f_r\|_{L_p}$ resp. $\|S_r\|_{L_p}$ which comes from the construction as given in the proof of Theorem 2. Note that there is no problem with $k \to -\infty$, the first step function in the construction may correspond to an arbitrarily large $k = k_0$. This expresses the fact that the systems under consideration do not form bases. The details are left to the reader.

Remark 5. For $p = 1$, Theorem 3 is in final shape: The system $\{\varphi_{k,\mathbf{i}}\}$ is a representation system in $L_1(\mathbf{R}^n)$ if and only if $\hat{\varphi}(\mathbf{0}) \neq 0$.

The situation is different for $p > 1$. From Theorem 1.7 of [BDR] it becomes clear (at least for $p = 2$) that some additional condition should be required. Unfortunately, we were not able to give the proof of Theorem 3 for the more general class of

$$\varphi \in \mathscr{L}_p(\mathbf{R}^n) = \left\{\psi \colon \sum_{\mathbf{i} \in \mathbf{Z}^n} |\psi(\mathbf{t} - \mathbf{i})| \in L_p([0, 1]^n)\right\}.$$

This class which is a subspace of $L_1 \cap L_p$, $1 < p < \infty$, was introduced in [JM] for studying L_p multiresolution analyses generated by refinable functions φ with noncompact support. The condition $\hat{\varphi}(\mathbf{0}) \neq 0$ which is clearly necessary if $p = 1$ but not for $p > 1$ (look at the Haar wavelet system on $\mathbf{R}^1$) also needs further elaboration.

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